Convolution of k-regular sequences

Clemens Heuberger   Daniel Krenn

July 31, 2023

Remark 0.1. For matrices A and B, let $A \otimes B$ denote their tensor product. Then for matrices A, A_0, A_1, B, B_0, B_1 of appropriate dimensions, we have

$$(A_0 \otimes B) + (A_1 \otimes B) = (A_0 + A_1) \otimes B$$
$$(A \otimes B_0) + (A \otimes B_1) = A \otimes (B_0 + B_1)$$

and

$$(A_0 \otimes B_0)(A_1 \otimes B_1) = (A_0 A_1) \otimes (B_0 B_1).$$

Note that for scalars a and b, we may identify $ab = (a) \otimes (b)$, where (a) and (b) are 1×1-matrices.

For a linear representation (u_X, X, w_X) of a k-regular sequence x, we denote its right vector-valued sequence by $v_X(n) \coloneqq \prod_{j=0}^{\ell-1} X(n_j) w_X$ if $n = (n_{\ell-1} \dots n_0)_k$ is the standard binary expansion of n. In particular, we have $x(n) = u_X^\top v_X(n)$ for all $n \geq 0$ and $v_X(0) = w_X$.

Theorem A. *Let x and y be k-regular sequences with linear representations (u_X, X, w_X) and (u_Y, Y, w_Y), respectively. Then the convolution*

$$z = \Bigl(\sum_{0 \leq j \leq n} x(j)\, y(n-j)\Bigr)_{n \geq 0}$$

of x and y is k-regular with linear representation

$$\left(\begin{pmatrix} u_X \otimes u_Y \\ 0 \\ 0 \end{pmatrix}, Z, \begin{pmatrix} w_X \otimes w_Y \\ 0 \\ 0 \end{pmatrix}\right)$$

satisfying

$$Z(0) = \begin{pmatrix} A_0 & B_0 & 0 \\ 0 & A_{k-1} & B_{k-1} \\ 0 & A_{k-2} & B_{k-2} \end{pmatrix}, \quad Z(1) = \begin{pmatrix} A_1 & B_1 & 0 \\ A_0 & B_0 & 0 \\ 0 & A_{k-1} & B_{k-1} \end{pmatrix}$$

and

$$Z(r) = \begin{pmatrix} A_r & B_r & 0 \\ A_{r-1} & B_{r-1} & 0 \\ A_{r-2} & B_{r-2} & 0 \end{pmatrix} \quad \textit{for all } r \geq 2,$$

with

$$A_r = \sum_{0 \leq s \leq r} (X(s) \otimes Y(r-s)), \quad B_r = \sum_{r < s < k} (X(s) \otimes Y(k+r-s))$$

for all $0 \leq r < k$.

Lemma 0.2. *Suppose we are in the set-up of Theorem A. Let* v_X *and* v_Y *be the right vector-valued sequences associated with the linear representations* (u_X, X, w_X) *and* (u_Y, Y, w_Y)*, respectively, and define*

$$v'(n) := \sum_{0 \leq j \leq n} v_X(j) \otimes v_Y(n-j).$$

Then

$$z = \Big((u_X^\top \otimes u_Y^\top) v'(n) \Big)_{n \geq 0}.$$

Proof. We have

$$\begin{aligned} z(n) &= \sum_{0 \leq j \leq n} u_X^\top v_X(j) u_Y^\top v_Y(n-j) = (u_X^\top \otimes u_Y^\top) \sum_{0 \leq j \leq n} (v_X(j) \otimes v_Y(n-j)) \\ &= (u_X^\top \otimes u_Y^\top) v'(n) \end{aligned}$$

which proves the lemma. □

Lemma 0.3. *Suppose we are in the set-up of Theorem A. The right vector-valued sequence* v' *of Lemma 0.2 satisfies*

$$v'(kn + r) = A_r v'(n) + B_r v'(n-1)$$

for all $n \geq 0$ *and* $0 \leq r < k$ *with*

$$A_r = \sum_{0 \leq s \leq r} (X(s) \otimes Y(r-s)), \quad B_r = \sum_{r < s < k} (X(s) \otimes Y(k+r-s))$$

for all $0 \leq r < k$ *and where* $v'(-1) = 0$ *because it is an empty sum.*

Proof. For $n \geq 0$ and $0 \leq r < k$, we obtain

$$\begin{aligned}
v'(kn+r) &= \sum_{0 \leq j \leq kn+r} v_X(j) \otimes v_Y(kn+r-j) \\
&= \sum_{0 \leq ki+s \leq kn+r} v_X(ki+s) \otimes v_Y(k(n-i)+r-s) \\
&= \sum_{\substack{0 \leq s \leq r \\ 0 \leq i \leq n}} v_X(ki+s) \otimes v_Y(k(n-i)+r-s) \\
&\quad + \sum_{\substack{r < s < k \\ 0 \leq i \leq n-1}} v_X(ki+s) \otimes v_Y(k(n-i-1)+k+r-s) \\
&= \sum_{\substack{0 \leq s \leq r \\ 0 \leq i \leq n}} (X(s)v_X(i)) \otimes (Y(r-s)v_Y(n-i)) \\
&\quad + \sum_{\substack{r < s < k \\ 0 \leq i \leq n-1}} (X(s)v_X(i)) \otimes (Y(k+r-s)v_Y(n-i-1)) \\
&= \sum_{0 \leq s \leq r} (X(s) \otimes Y(r-s)) \sum_{0 \leq i \leq n} (v_X(i) \otimes v_Y(n-i)) \\
&\quad + \sum_{r < s < k} (X(s) \otimes Y(k+r-s)) \sum_{0 \leq i \leq n-1} (v_X(i) \otimes v_Y(n-i-1)) \\
&= \sum_{0 \leq s \leq r} (X(s) \otimes Y(r-s))v'(n) + \sum_{r < s < k} (X(s) \otimes Y(k+r-s))v'(n-1).
\end{aligned}$$

□

Proposition 0.4. *Let $z = (u'v'(n))_{n \geq 0}$ be a k-regular sequence with left vector u' and right vector-valued sequence v' satisfying*

$$v'(kn+r) = A_r v'(n) + B_r v'(n-1)$$

for all $n \geq 0$ and $0 \leq r < k$ with some matrices A_r and B_r for all $0 \leq r < k$.

Then z has a linear representation

$$\left(\begin{pmatrix} u' \\ 0 \\ 0 \end{pmatrix}, Z, \begin{pmatrix} v'(0) \\ 0 \\ 0 \end{pmatrix} \right)$$

with right vector-valued sequence $v_Z(n) = (v'(n), v'(n-1), v'(n-2))^\top$ and satisfying

$$Z(0) = \begin{pmatrix} A_0 & B_0 & 0 \\ 0 & A_{k-1} & B_{k-1} \\ 0 & A_{k-2} & B_{k-2} \end{pmatrix}, \quad Z(1) = \begin{pmatrix} A_1 & B_1 & 0 \\ A_0 & B_0 & 0 \\ 0 & A_{k-1} & B_{k-1} \end{pmatrix}$$

and

$$Z(r) = \begin{pmatrix} A_r & B_r & 0 \\ A_{r-1} & B_{r-1} & 0 \\ A_{r-2} & B_{r-2} & 0 \end{pmatrix} \quad \textit{for all } r \geq 2.$$

Proof. The result is a straight-forward generalization (from scalar coefficients to our matrix-valued coefficients A_r and B_r) of a special case of [2, Theorem A]. It can also be easily verified by a direct computation. □

Proof of Theorem A. The convolution z of the two k-regular sequences x and y is indeed again k-regular; see [1, Theorem 3.1]. The linear representation follows by combining Lemma 0.2, Lemma 0.3 and Proposition 0.4. □

References

[1] Jean-Paul Allouche and Jeffrey Shallit, *The ring of k-regular sequences*, Theoret. Comput. Sci. **98** (1992), no. 2, 163–197. MR 1166363

[2] Clemens Heuberger, Daniel Krenn, and Gabriel F. Lipnik, *Asymptotic analysis of q-recursive sequences*, Algorithmica **84** (2022), no. 9, 2480–2532. MR 4467813